

# An axiomatic approach to finite means

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## Abstract

In this paper we analyze the notion of a finite mean from an axiomatic point of view. We discuss several axiomatic alternatives, with the aim of establishing a universal definition reconciling all of them and exploring theoretical links to some branches of Mathematics as well as to multidisciplinary applications.

*Keywords:* Means; Axiomatics; Iterativity; Totally ordered sets; Lattices; Semigroups; Topological spaces; Binary operations; Functional equations; Social choice; Ranking sets of objects; Fuzzy set theory.

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## 1. Introduction

The search for a suitable definition of a mean is indeed an old question (see e.g. [44, 6, 28, 16, 31, 24]), on which we want to analyze new trends and possibilities, coming from several branches of Mathematics (see e.g. [40, 41, 23]) as well as from a miscellaneous wide set of interdisciplinary applications (e.g.: aggregation of individual preferences into a social one, in Mathematical Social Choice, see [20, 15, 17, 29, 14], or the study of aggregation operators in Fuzzy Set Theory, see [26, 46, 22, 4, 13, 35]).

Several miscellaneous examples of contexts where some kind of a mean plays a crucial role are: descriptive statistical studies, where centralization measures (means of  $n$  real numbers, in particular) as well as dispersion measures constitute a key notion; ordered sets; topological spaces where a continuous topological mean can be defined (see e.g. [6]); divisible groups, where an algebraic mean makes sense (see e.g. [40]); sets of profiles of individual preferences, arising in Social Choice; extensions of orderings from a set to its power set following

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different criteria (see e.g. [7]); aggregation operators arising in fuzzy set theory (e.g.: triangular norms and conorms, fuzzy integrals, ordered weighted functions and so on, see e.g. [26, 52, 45, 8, 9, 27]).

One may wonder why a so high number of different definitions are encountered in the literature. At this extent, we should notice that, depending on the contexts, there are operations that can not be done, or particular features that can not be assumed a priori. Notice, as an evident example, that the definition of a mean on  $n$ -real numbers (where we can perform algebraic operations, there is a linear order, etc.) could actually be much more demanding than the notion of an  $n$ -mean on an abstract set where there is no structure (neither algebraic nor topological) given a priori. In a sense, each context has a strong influence on the definition of an  $n$ -mean that we may consider on it.

A glance at the jungle of (existing and possibly new) definitions of the concept of a mean naturally suggests to try putting some order in the field, by showing links and hierarchies among the corresponding contexts and definitions, and finally looking for a unified theory.

In our approach we will consider a nonempty set  $X$ , as well as the set  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$  and a map  $F : \mathcal{X} \rightarrow X$ , studying the restrictions  $F_n$  of  $F$  to each  $X^n$ . Having this in mind, we will talk about finite means. (See Remark 3.1 (ii) in Section 3). Only finite means will be considered in this paper.

The structure of the paper goes as follows. After a section of Preliminaries in which we discuss some known definitions of an  $n$ -mean in different contexts, in Section 3 we introduce some systems of axioms to define means, starting with the classical definitions on the real line and paying a particular attention to the classical Kolmogorov's approach. Then we extend the study to totally ordered sets, lattices, and other topological or algebraic settings. In Section 4 we study additional properties of means, using iterativity and some functional equations involved. In Section 5 we describe miscellaneous applications in different contexts. To conclude, Sections 6 and 7 discuss the results obtained and contain further comments and remarks.

## 2. Preliminaries

In this section we recall different definitions of the concept of an  $n$ -mean, which have already appeared in various mathematical contexts. Then we discuss the hierarchy among different definitions, and recall some central results.

### 2.1. Towards a definition of a mean for $n$ elements

The idea of a mean for  $n$  elements that belong to a given set (henceforward, an  $n$ -mean) naturally depends on the *context* involved. Thus, it can be understood from the point of view of *order* as suitable choices, made on finite subsets with  $n$  elements of a set endowed with some ordering (e.g.: a lattice). From the point of view of *topology* we may think about a suitable continuous map defined on the Cartesian product of  $n$  copies of a topological space  $(X, \tau)$ , and taking values on  $X$ . From an algebraic approach, we may consider some  $n$ -ary

operation defined on a set. Working in the framework of functional equations, an  $n$ -mean could be interpreted as a solution of a suitable set of simultaneous functional equations (e.g.: anonymity). Needless to say, many other possible approaches might be explored.

In this section, we give an account of some of the typical definitions of an  $n$ -mean, arising in different settings. Bearing these definitions in mind, in next Section 3 we will introduce several notions of finite means, following the presentation and relationship of these approaches introduced now, and looking for a universal definition valid for all of them. This will finally lead to the concept of a general mean. (See Definition 3.39).

To start with, we analyze different possible definitions of the notion of an  $n$ -mean, which arise when dealing with contexts related to *Order*.

**Definition 2.1.** Let  $(L, \vee, \wedge)$  be a nonempty lattice (i.e., a partially ordered set  $(L, \preceq)$  with the property that any two elements  $x, y \in L$  have both a supremum  $x \vee y$  and an infimum  $x \wedge y$  in  $(L, \preceq)$ ). The binary operation  $\vee$  (respectively,  $\wedge$ ) is usually called the *join* (respectively, the *meet*). An  $n$ -mean is a map  $F : L^n \rightarrow L$  such that  $y_1 \wedge \dots \wedge y_n \preceq F(y_1, \dots, y_n) \preceq y_1 \vee \dots \vee y_n$ , for every  $(y_1, \dots, y_n) \in L^n$ .

**Remark 2.2.** This Definition 2.1 is an extension of a very old one (see Definition 2.3). A very important particular case of Definition 2.1 is Definition 2.5 (see below), provided that the nonempty set is endowed with a total order.

**Definition 2.3.** Let  $A \subseteq \mathbb{R}$  be a nonempty subset of real numbers. An  $n$ -mean is a map  $F : A^n \rightarrow A$  such that  $\min\{a_1, \dots, a_n\} \leq F(a_1, \dots, a_n) \leq \max\{a_1, \dots, a_n\}$ , for every  $(a_1, \dots, a_n) \in A^n$ .

**Remark 2.4.** This Definition 2.3 is indeed very old. It dates back to A.L. Cauchy, who introduced it in 1821. (See [19]). The property involved in it is usually called *internality* (see e.g. [36]).

**Definition 2.5.** Let  $X$  be a nonempty set endowed with a total order  $\preceq$  (i.e.: the binary relation  $\preceq$  defined on  $X$  is antisymmetric, transitive and total). An  $n$ -mean is a map  $F : X^n \rightarrow X$  such that  $\min\{x_1, \dots, x_n\} \preceq F(x_1, \dots, x_n) \preceq \max\{x_1, \dots, x_n\}$ , for every  $(x_1, \dots, x_n) \in X^n$ .

Now we recall a classical definition of an  $n$ -mean naturally arising in *Topology*.

**Definition 2.6.** (Aumann [6]; Chichilnisky [20])

Let  $(X, \tau)$  be a topological space (i.e.:  $X$  is a nonempty set endowed with a topology  $\tau$ ). A *topological  $n$ -mean* is an  $n$ -variate map  $F : X^n \rightarrow X$  such that the following conditions hold:

- (i) (Anonymity-neutrality)  $F(x_1, \dots, x_n) = F(x_{\sigma(1)}, \dots, x_{\sigma(n)})$  holds true for any  $(x_1, \dots, x_n) \in X^n$  and every permutation  $\sigma$  of the set  $\{1, \dots, n\}$ .
- (ii) (Unanimity)  $F(t, \dots (n \text{ times}) \dots, t) = t$ , for every  $t \in X$ .

- (iii) (Continuity)  $F$  is a continuous map, considering the topology  $\tau$  on  $X$  and the corresponding product topology on  $X^n$ .

**Remark 2.7.** The property (i) of anonymity-neutrality is also known as *symmetry* in the specialized literature, whereas the property (ii) of unanimity is sometimes called *idempotence* (see e.g. [36]).

Now, we examine some definitions of the notion of an  $n$ -mean, coming from *Algebra*.

**Definition 2.8.** Let  $(S, *)$  be a semigroup (i.e.:  $S$  is a nonempty set with an associative binary operation  $*$ ). A *semigroup  $n$ -mean* is a map  $F : S^n \rightarrow S$  such that  $s_1 * \dots * s_n = F(s_1, \dots, s_n) * \dots (n\text{-times}) \dots * F(s_1, \dots, s_n)$  for any  $(s_1, \dots, s_n) \in S^n$ .

**Definition 2.9.** (Candea and Induráin [16]) Let  $G$  be a nonempty set endowed with a binary operation  $*$ . An *algebraic  $n$ -mean* is an  $n$ -ary operation  $F : G^n \rightarrow G$  such that the following conditions hold:

- (i) (Anonymity-neutrality)  $F(g_1, \dots, g_n) = F(g_{\sigma(1)}, \dots, g_{\sigma(n)})$  holds for every  $(g_1, \dots, g_n) \in G^n$  and every permutation  $\sigma$  of the set  $\{1, \dots, n\}$ .
- (ii) (Unanimity)  $F(t, \dots (n\text{-times}) \dots, t) = t$ , for every  $t \in G$ .
- (iii) (Algebraic stability)  $F(g_1 * h_1, \dots, g_n * h_n) = F(g_1, \dots, g_n) * F(h_1, \dots, h_n)$ , for every  $(g_1, \dots, g_n), (h_1, \dots, h_n) \in G^n$ .

It is plain that Definition 2.3 is a particular case of Definition 2.5. Besides, Definition 2.5 is a particular case of Definition 2.1.

In addition,  $n$ -means in the sense of Definitions 2.3 to 2.1 satisfy the condition of unanimity stated in Definitions 2.6, 2.8 and 2.9. However,  $n$ -means in the sense of Definitions 2.3 to 2.1 *do not satisfy, in general, the condition of anonymity-neutrality*. Observe that in Definitions 2.3 to 2.1 the order of the elements in an  $n$ -tuple could be relevant.

Now we mention some deeper results, also related to the previous definitions.

**Theorem 2.10.** (Eckmann [28]) *Let  $(Y, \tau)$  be a connected topological space. If  $\Pi_k(Y)$  stands for the  $k$ -th homotopy group of  $Y$ , the existence of a topological  $n$ -mean on  $Y$  immediately induces an algebraic  $n$ -mean on  $\Pi_k(Y)$ .*

Similar results to that furnished by Theorem 2.10 hold if we consider homology or cohomology groups instead of homotopy groups.

**Theorem 2.11.** (Keesling [41]) *A connected, compact and Hausdorff topological group  $(G, *, \tau)$  admits a topological  $n$ -mean if and only if it admits an algebraic  $n$ -mean.*

**Theorem 2.12.** (see e.g. Candea and Induráin [16]) *Let  $(G, *)$  be a group, and suppose that an algebraic  $n$ -mean  $F : G^n \rightarrow G$  exists. Then, the group  $G$  must be Abelian (i.e.:  $x * y = y * x$ , for every  $x, y \in X$ ) and  $n$ -divisible (i.e., for every  $x \in G$  there exists a unique element  $y \in G$  such that  $y^n = y * \dots (n\text{-times}) \dots * y = x$ ). Moreover,  $F$  is also a semigroup  $n$ -mean.*

**Theorem 2.13.** (Aumann [6]; Chichilnisky [20]) *Let  $(X, \tau)$  be a finite CW-complex. Then  $X$  admits a topological  $n$ -mean for every  $n$  if and only if  $(X, \tau)$  is contractible. The  $n$ -means here can be interpreted as retractions of the arithmetic mean defined on some Euclidean space  $\mathbb{R}^k$ , such that  $X \subseteq \mathbb{R}^k$ .*

**Theorem 2.14.** (see Candéal and Induráin [16]) *Let  $(G, *)$  be a group. If  $G$  admits an algebraic  $n$ -mean for every  $n$ , then  $(G, *)$  is isomorphic to a vector space over the field  $(\mathbb{Q}, +, \cdot)$  of rational numbers. In particular, if  $(G, *)$  is not trivial it must, a fortiori, be infinite.*

## 2.2. From bivariate means to $n$ -variate means

We wonder if we can get  $n$ -variate means from  $m$ -variate ones, if  $n \neq m$ . In particular, we ask ourselves about the possibility of defining  $n$ -means for any  $n \geq 2$  just from bivariate means, using some suitable procedure, when possible. A crucial particular case, namely iterativity, will be analyzed in Section 4 for general finite means.

Here we introduce a list of miscellaneous results relating  $n$ -means and  $m$ -means provided that  $n \neq m$ .

**Proposition 2.15.** *In the context of Definitions 2.6 to 2.9, the existence of an  $n$ -mean  $F_n$  implies the existence of an  $m$ -mean  $F_m$  provided that  $m$  divides  $n$ . Also, the existence of an  $n$ -mean  $F_n$  and a  $k$ -mean  $F_k$  with  $k = \frac{m!}{n! \cdot (m-n)!}$  implies the existence of an  $m$ -mean.*

*Proof.* To prove the first fact, if  $n = k \cdot m$  we may define  $F_m(x_1, \dots, x_m) = F_n(x_1, \dots, x_m, \dots, x_1, \dots, x_m)$  (repeated  $k$  times) for any  $(x_1, \dots, x_m) \in X^m$ . To prove the second one, for any  $(x_1, \dots, x_m)$ , we define  $F_m(x_1, \dots, x_m) = F_k(F_n(\bar{a}_1), \dots, F_n(\bar{a}_k))$ , where  $\{\bar{a}_1, \dots, \bar{a}_k\}$  is the family of all the combinations of  $n$  elements taken from the set  $\{x_1, \dots, x_m\}$ .  $\square$

**Theorem 2.16.** (see Campión et al. [14]) *Let  $X$  be a nonempty set. Suppose that  $X$  admits a 2-mean  $F : X^2 \rightarrow X$  in some of the contexts of Definitions 2.5, 2.7 and 2.8. Assume in addition that  $F$  is associative, that is  $F(x, F(y, z)) = F(F(x, y), z)$  for every  $x, y, z \in X$ . Then  $F$  induces an  $n$ -mean  $F_n$  for every  $n \in \mathbb{N}$  in the corresponding context, through the recurrence  $F_n(x_1, \dots, x_n) = F(F_{n-1}(x_1, \dots, x_{n-1}), x_n)$   $((x_1, \dots, x_n) \in X^n)$ .*

Moreover, in the context of Theorem 2.16,  $F$  can be viewed here as an *algebraic binary operation*  $*$  defined by  $F(x, y) = x * y$ , for every  $x, y \in X$ . This operation  $*$  is associative, commutative, and such that every element is idempotent. Indeed,  $*$  is a semilatticial operation, i.e.:  $*$  acts as the join operation of a lattice. Notice also that the  $n$ -means so obtained can directly be expressed in terms of the 2-mean we had at hand.

## 3. Axiomatics for finite means

Before introducing several sets of axioms, we discuss how the axioms should be, paying attention to several remarkable aspects (e.g.: the possibility of reckoning means through a computer algorithm).

### 3.1. Some helpful results based on functional equations

Let  $X$  be a nonempty set. Roughly speaking, we understand an  $n$ -mean as a map  $F_n : X^n \rightarrow X$ . Thus, we intend to define those  $n$ -means for every  $n$ , looking for some sort of compatibility between  $n$ -means and  $m$ -means if  $n \neq m$ . Bearing in mind some idea of recurrence or compatibility, so that we could somehow use the computation of an  $n$ -mean to that of an  $m$ -mean if  $n < m$ , we will directly consider maps defined on  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$  and taking values on  $X$ .

**Remarks 3.1.** (i) The idea of working directly on  $\mathcal{X}$  is not new. It comes back at least to 1930, when A. Kolmogorov introduced a system of axioms to define a mean in the real line  $\mathbb{R}$ , disregarding the amount of numbers involved. (See [44]). Many other authors have used this idea in recent works. (See e.g. [32, 35, 18]).

(ii) The term *finite* is used here to distinguish this approach to other ones that involve an infinite set of elements of which a certain mean or average value is computed. (A typical example is the concept of gravity center of a body, arising in Physics).

In this direction, first we introduce some definitions.

**Definition 3.2.** Let  $X$  be a nonempty set and  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$ . Consider a map  $F : \mathcal{X} \rightarrow X$ . Let  $F_n$  be the restriction of  $F$  to  $X^n$ . The map  $F$  is said to satisfy the condition of:

- (i) *anonymity-neutrality*<sup>1</sup>, if for every  $n \in \mathbb{N}$  and  $(x_1, \dots, x_n) \in X^n$  it holds true that  $F_n(x_1, \dots, x_n) = F_n(x_{\sigma(1)}, \dots, x_{\sigma(n)})$  for any permutation  $\sigma$  of the set  $\{1, \dots, n\}$ ,
- (ii) *unanimity*, if for every  $n \in \mathbb{N}$  and  $x \in X$  it holds true that  $F_n(x, \dots, x) = x$ ,
- (iii) *compatibility*, if for any  $m, n \in \mathbb{N}$  and  $(x_1, \dots, x_{m+n}) \in X^{m+n}$ , it holds that  $F_{m+n}(x_1, \dots, x_{m+n}) = F_{m+n}(\bar{x}, \dots, (m \text{ times}) \dots, \bar{x}, x_{m+1}, \dots, x_{m+n})$ , where  $\bar{x} = F_m(x_1, \dots, x_m)$ ,
- (iv) *generalized bisymmetry*<sup>2</sup>, if for every  $(x_1, \dots, x_k, x_{11}, \dots, x_{nn}) \in X^{n^2+k}$ , it holds that  $F_{n+k}(F_n(x_{11}, \dots, x_{1n}), \dots, F_n(x_{n1}, \dots, x_{nn}), x_1, \dots, x_k) = F_{n+k}(F_n(x_{11}, \dots, x_{n1}), \dots, F_n(x_{1n}, \dots, x_{nn}), x_1, \dots, x_k)$ ,
- (v) *associativity*, if for every  $x_1, x_2, x_3 \in X$  it holds that  $F_2(x_1, F_2(x_2, x_3)) = F_2(F_2(x_1, x_2), x_3)$ ,

<sup>1</sup>As already commented in Remark 2.7, this property is also called *symmetry* by some authors.

<sup>2</sup>A function  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  is said to be *bisymmetric* if  $F(F(a, b), F(c, d)) = F(F(a, c), F(b, d))$  holds true for every  $a, b, c, d \in \mathbb{R}$ .

- (vi) *selection*, if for every  $n \in \mathbb{N}$  and  $(x_1, \dots, x_n) \in X^n$  it holds true that  $F_n(x_1, \dots, x_n) \in \{x_1, \dots, x_n\}$ ,
- (vii) *replicability*, if  $F_1(x) = x$  for every  $x \in X$  and, in addition, for every  $n, k \in \mathbb{N}$  and  $(x_1, \dots, x_n) \in X^n$  it holds true that  $F_n(x_1, \dots, x_n) = F_{nk}(x_1, \dots (k \text{ times}) \dots, x_1, \dots, x_n, \dots (k \text{ times}) \dots, x_n)$ .

**Theorem 3.3.** *Let  $X$  be a nonempty set,  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$ . Let  $F : \mathcal{X} \rightarrow X$  be a map that satisfies anonymity-neutrality. The following statements are equivalent:*

- (i)  *$F$  satisfies replicability and compatibility.*
- (ii)  *$F$  satisfies unanimity and generalized bisymmetry.*
- (iii)  *$F$  satisfies unanimity and compatibility.*
- (iv)  *$F$  satisfies replicability and generalized bisymmetry.*

*Proof.* We will follow the schema: (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (i).

First we prove that (i)  $\Rightarrow$  (ii):

Let  $\bar{z} = (F_n(x_{11}, \dots, x_{1n}), \dots, F_n(x_{n1}, \dots, x_{nn}), x_1, \dots, x_k) \in X^{n+k}$ . By replicability, we have that  $F_{n+k}(\bar{z}) = F_{n^2+kn}(\bar{z}, \dots (n^2+kn \text{ times}) \dots, \bar{z})$ . Calling  $\bar{x}_i = F_n(x_{i1}, \dots, x_{in})$  ( $1 \leq i \leq n$ ), and using systematically the hypotheses of anonymity-neutrality and compatibility, it follows that:  $F_{n^2+kn}(\bar{z}, \dots (n^2+kn \text{ times}) \dots, \bar{z}) = F_{n^2+kn}(\bar{x}_1, \dots (n \text{ times}) \dots, \bar{x}_1, \dots, \bar{x}_n, \dots (n \text{ times}) \dots, \bar{x}_n, x_1 \dots (n \text{ times}) \dots, x_1, \dots, x_k, \dots (n \text{ times}) \dots, x_k) = F_{n^2+kn}(x_{11}, \dots, x_{1n}, \dots, x_{n1}, \dots, x_{nn}, x_1 \dots (n \text{ times}) \dots, x_1, \dots, x_k, \dots (n \text{ times}) \dots, x_k)$ .

In the same way, if  $\bar{t} = (F_n(x_{11}, \dots, x_{n1}), \dots, F_n(x_{1n}, \dots, x_{nn}), x_1, \dots, x_k)$ , we will also arrive at  $F_{n+k}(\bar{t}) = F_{n^2+kn}(x_{11}, \dots, x_{n1}, \dots, x_{1n}, \dots, x_{nn}, x_1, \dots (n \text{ times}) \dots, x_1, \dots, x_k, \dots (n \text{ times}) \dots, x_k) = F_{n^2+kn}(x_{11}, \dots, x_{1n}, \dots, x_{n1}, \dots, x_{nn}, x_1, \dots (n \text{ times}) \dots, x_1, \dots, x_k, \dots (n \text{ times}) \dots, x_k)$ .

Then we prove the implication (ii)  $\Rightarrow$  (iii), where the assumption of anonymity-neutrality is not necessary:

Let  $m, n \in \mathbb{N}$ . Let  $(x_1, \dots, x_{m+n}) \in X^{m+n}$ . Then, by the unanimity condition, we get that  $F_{m+n}(x_1, \dots, x_{m+n}) = F_{n+m}(F_n(x_1, \dots (n \text{ times}) \dots, x_1), \dots, F_n(x_n, \dots (n \text{ times}) \dots, x_n), x_{n+1}, \dots, x_{n+m})$ . Using now the generalized bisymmetry condition, we finally arrive at  $F_{n+m}(F_n(x_1, \dots, x_n), \dots (n \text{ times}) \dots, F_n(x_1, \dots, x_n), x_{n+1}, \dots, x_{n+m})$ .

Now we prove that (iii)  $\Rightarrow$  (iv): On the one hand, given  $x \in X$ ,  $F_1(x) = x$  follows from unanimity. On the other hand, by anonymity-neutrality and compatibility, given  $n, k \in \mathbb{N}$  and  $(x_1, \dots, x_n) \in X^n$  it follows that  $F_{nk}(x_1, \dots (k \text{ times}) \dots, x_1, \dots, x_n, \dots (k \text{ times}) \dots, x_n) = F_{nk}(x_1, \dots, x_n, \dots (k \text{ times}) \dots, x_1, \dots, x_n) = F_{nk}(F_n(x_1, \dots, x_n), \dots (nk \text{ times}) \dots, F_n(x_1, \dots, x_n))$ . Then, by unanimity, this yields  $F_n(x_1, \dots, x_n)$ .

Finally, the implication (iv)  $\Rightarrow$  (i) is entirely analogous to (ii)  $\Rightarrow$  (iii).

This finishes the proof.  $\square$

**Remark 3.4.** The conditions in Definition 3.2 depend on functional equations. This is the approach followed by some authors as, e.g., J. Aczél in [1].

Now we introduce a new definition for the particular case in which the set  $X$  is endowed with a total (also known as linear) order  $\preceq$ . In this case, some kind of compatibility between the order  $\preceq$  and the mean is usually demanded. This gives rise, in particular, to different notions of monotonicity that play a crucial role in this setting. (See e.g. next subsection 3.2, devoted to analyze the Kolmogorov's approach to define finite means on real numbers).

**Definition 3.5.** Let  $X$  be a nonempty set endowed with a total order  $\preceq$ . Let  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$ . A map  $F : \mathcal{X} \rightarrow X$  is said to satisfy the condition of:

- (viii) *monotonicity*, if the restriction of  $F$  to  $X^n$ , that we denote  $F_n$ , is monotone in each variable as regards  $\preceq$ , for every natural number  $n$ : that is,  $x_1 \preceq y_1, \dots, x_n \preceq y_n \Rightarrow F_n(x_1, \dots, x_n) \preceq F_n(y_1, \dots, y_n)$  holds for every  $(x_1, \dots, x_n, y_1, \dots, y_n) \in X^{2n}$ .
- (ix) *strict monotonicity*, if the restriction of  $F$  to  $X^n$ , that we denote  $F_n$ , is strictly monotone in each variable as regards  $\preceq$ , for all  $n \in \mathbb{N}$ : that is,  $x_1 \preceq y_1, \dots, x_n \preceq y_n \Rightarrow F_n(x_1, \dots, x_n) \preceq F_n(y_1, \dots, y_n)$  holds for every  $(x_1, \dots, x_n, y_1, \dots, y_n) \in X^{2n}$ , and if in addition  $(x_1, \dots, x_n) \neq (y_1, \dots, y_n)$ , then  $F_n(x_1, \dots, x_n) \neq F_n(y_1, \dots, y_n)$  holds, too.
- (x) *internality*, when, for any  $n \in \mathbb{N}$ , and any  $(x_1, \dots, x_n) \in X^n$ , the restriction of  $F$  to  $X^n$ , that we denote  $F_n$ , accomplishes that  $\min\{x_1, \dots, x_n\} \preceq F_n(x_1, \dots, x_n) \preceq \max\{x_1, \dots, x_n\}$ . Here the minima and maxima are taken as regards the total order  $\preceq$ .

The proof of next Theorem 3.6 is similar to that of Proposition 2.54 in [36]. We just include it here with the aim of providing an easier reading.

**Theorem 3.6.** Let  $X$  be a nonempty set endowed with a total order  $\preceq$ . Let  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$ . Let  $F : \mathcal{X} \rightarrow X$  be a map that satisfies monotonicity. Then  $F$  satisfies unanimity if and only if it satisfies internality.

*Proof.* Again, let  $F_n$  stand for the restriction of  $F$  to  $X^n$ . First notice that if  $F$  satisfies monotonicity and unanimity, then, given  $(a_1, \dots, a_n) \in X^n$  and  $a = \min\{a_1, \dots, a_n\}$ ;  $b = \max\{a_1, \dots, a_n\}$ , we have that  $a = F_n(a, \dots, (n \text{ times}) \dots, a) \preceq F_n(a_1, \dots, a_n) \preceq F_n(b, \dots, (n \text{ times}) \dots, b) = b$ . Conversely, if  $F$  satisfies internality and monotonicity, then  $x = \min\{x, \dots, (n \text{ times}) \dots, x\} \preceq F(x, \dots, (n \text{ times}) \dots, x) \preceq \max\{x, \dots, (n \text{ times}) \dots, x\} = x$ . So  $F(x, \dots, (n \text{ times}) \dots, x) = x$  holds for every  $x \in X$ .  $\square$

### 3.2. Kolmogorov's approach

Now we pass to discuss some aspects of the classical Kolmogorov's setting to define means on the real line  $\mathbb{R}$ .

Endowing the real line  $\mathbb{R}$  with the usual Euclidean topology  $\tau_u$ , we may introduce an idea of continuity for means defined on  $\mathbb{R}$ .



**Definition 3.7.** Let  $\mathcal{R} = \bigcup_{n=1}^{\infty} \mathbb{R}^n$ . A function  $M : \mathcal{R} \rightarrow \mathbb{R}$  is said to satisfy the condition of

- (xi) *continuity*, if, for every  $n \in \mathbb{N}$ , the restriction of  $M$  to  $\mathbb{R}^n$  is a continuous function with respect to the usual topology on the real line, and the corresponding product topology on  $\mathbb{R}^n$ .

**Definition 3.8.** (Kolmogorov, 1930 [44]) Let  $\mathcal{R} = \bigcup_{n=1}^{\infty} \mathbb{R}^n$ . A function  $M : \mathcal{R} \rightarrow \mathbb{R}$  is said to be a *regular mean on  $\mathbb{R}$*  if it satisfies the conditions of anonymity-neutrality, unanimity, compatibility, strict monotonicity (with respect to the usual order  $\leq$  on  $\mathbb{R}$ ), and continuity. (In Kolmogorov's notation,  $M$  is a sequence  $(M_n)_{n=1}^{\infty}$  of functions  $M_n : \mathbb{R}^n \rightarrow \mathbb{R}$ .)

**Example 3.9.** For any  $n \in \mathbb{N}$ , define  $m_n : \mathbb{R}^n \rightarrow \mathbb{R}$  by  $m_n(x_1, \dots, x_n) = \min\{x_1, \dots, x_n\}$ , for all  $(x_1, \dots, x_n) \in \mathbb{R}^n$ . Let  $M : \mathcal{R} \rightarrow \mathbb{R}$  be defined as the sequence of functions  $M = (m_n)_{n=1}^{\infty}$ . It is straightforward to see that  $M$  satisfies the conditions of anonymity-neutrality, unanimity, compatibility, monotonicity, and continuity. But it fails to be a regular mean, since it is not strictly monotone. To see this just observe that  $m_2(0, 1) = 0 = m_2(0, 2)$ , whereas  $(0, 1) \neq (0, 2)$ .

**Theorem 3.10.** (Kolmogorov, 1930 [44]; Nagumo, 1930 [48]; Aczél, 1948 [1]) *If  $M = (M_n)_{n=1}^{\infty}$  is a regular mean, then there exists a continuous and strictly increasing function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $M_n(a_1, \dots, a_n) = f^{-1}[\frac{f(a_1) + \dots + f(a_n)}{n}]$  holds true for every  $n \in \mathbb{N}$  and  $(a_1, \dots, a_n) \in \mathbb{R}^n$ .*

**Remarks 3.11.**

- (i) In an independent way, and simultaneously to Kolmogorov's work, M. Nagumo also introduced a system of axioms for means involving real numbers. (See [48]). That axiomatics is almost identical to Kolmogorov's one. The main difference is that Kolmogorov worked on the real line, and Nagumo worked on a real interval. In fact, the regular means given as stated in Theorem 3.10 are usually known as *Nagumo-Kolmogorov means*.

Kolmogorov defined a mean as a map  $M : \bigcup_{n=1}^{\infty} \mathbb{R}^n \rightarrow \mathbb{R}$  that obeys the following axioms: K1)  $M$  is continuous and increasing in each variable; K2)  $M$  is symmetric (i.e., it satisfies anonymity-neutrality); K3)  $M(x, \dots, x) = x$ ; K4)  $M(x_1, \dots, x_n) = M(M(x_1, \dots, x_r), \dots (r \text{ times}) \dots, M(x_1, \dots, x_r), x_{r+1}, \dots, x_n)$  for any  $1 \leq r \leq n$ .

Nagumo defined a mean as a function  $\mu : \bigcup_{n=1}^{\infty} [a, b]^n \rightarrow [a, b]$  accomplishing the following conditions: N1)  $\mu$  satisfies anonymity-neutrality; N2)  $\mu(x_1, \dots, x_n) = \mu(\mu(x_1, \dots, x_r), \dots (r \text{ times}) \dots, \mu(x_1, \dots, x_r, x_{r+1}, \dots, x_n))$  for any  $1 \leq r \leq n$ ; N3)  $\mu$  is continuous; N4) if  $x_1 < x_2$ , then  $x_1 < \mu(x_1, x_2) < x_2$ ; N5)  $\mu(a, \dots, a) = a$ .

Comparing both systems of axioms, we see that K1 is equivalent to N3 plus N4. Also, K2 (respectively, K3, K4) is identical to N1 (respectively, to N5, N2).

- (ii) Kolmogorov's approach was then extended to the infinite case, to deal with distributions of probability, by B. De Finetti in 1931. (See [30, 47]).
- (iii) Theorem 3.10 still remains true if we adopt Definition 3.8 to similarly define the concept of a regular mean  $(M_n)_{n=1}^\infty$  on the set  $\bigcup_{n=1}^\infty J^n$ , where  $J$  is an interval of  $\mathbb{R}$ . The functions  $M_n$  go now from  $J^n$  into  $J$ . A typical situation, encountered in the study of aggregation operators in fuzzy set theory, appears when  $J = [0, 1]$ . (See e.g. [36, 35]).
- (iv) Regular means can be used to define suitable means on topological spaces that are homeomorphic to  $\mathbb{R}$ , as follows: Let  $(X, \tau)$  be a topological space, homeomorphic to the real line endowed with the usual topology  $\tau_u$ . Let  $H : (X, \tau) \rightarrow (\mathbb{R}, \tau_u)$  be an homeomorphism. Let  $M = (M_n)_{n=1}^\infty$  be a regular mean on the real line. Given  $n \in \mathbb{N}$ , define the map  $F_n : X^n \rightarrow X$  as  $F_n(x_1, \dots, x_n) = H^{-1}(M_n(H(x_1), \dots, H(x_n)))$  for any  $(x_1, \dots, x_n) \in X^n$ . The sequence of maps  $(F_n)_{n=1}^\infty$  can be now considered as a "regular mean" on  $X$ . It is straightforward to see that the analogous of the properties in Definition 3.10 are now accomplished by the function  $F = (F_n)_{n=1}^\infty$ , defined on  $\mathcal{X} = \bigcup_{n=1}^\infty X^n$  and taking values on  $X$ , provided that on  $X^n$  we consider the product topology induced by  $\tau$ , and  $X$  is endowed with the linear order  $\preceq$  given by  $x \preceq y \Leftrightarrow H(x) \leq H(y)$  ( $x, y \in X$ ).

### 3.3. Finite means on totally ordered sets

Let us analyze now what could happen if we adopt Kolmogorov's axioms to work in a more general setting. Instead of dealing with real numbers (and consequently with the set  $\mathcal{R}$ ), suppose that we start with an abstract nonempty set  $X$  endowed with a total order  $\preceq$ . Obviously, we cannot expect an analogous of Theorem 3.10 to be true here, since operations as addition or dividing by a natural number  $n$  are not defined a priori on the abstract set  $X$ . Moreover, to handle some kind of continuity,  $X$  should also be endowed with a topology.

However, depending on the topology considered on a nonempty set  $X$ , it could happen that no regular mean satisfying Kolmogorov's axioms exists on  $X$ . In fact, as commented in Theorem 2.13 above, continuity can be incompatible with anonymity-neutrality plus unanimity. This happens, for instance, in a special kind of topological spaces known in the literature as non-contractible finite cellular CW-complexes<sup>3</sup>. (See [6, 20]).

To start with, we may define a mean "à la Kolmogorov" on a nonempty set endowed with a total order  $\preceq$ , as follows:

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<sup>3</sup>Roughly speaking, a CW-complex is made of cells, such that each of them is homeomorphic to the interior of a sphere in the Euclidean  $k$ -dimensional space  $\mathbb{R}^k$ , for some  $k \in \mathbb{N}$ . The precise definition prescribes how the cells may be topologically glued together. The C stands for "closure-finite", and the W for "weak topology". They were introduced by J. H. C. Whitehead to meet the needs of homotopy theory. In addition, contractibility means that the whole space could be continuously deformed to become a single point.

**Definition 3.12.** Let  $X$  be a nonempty set endowed with a total order  $\preceq$ . Let  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$ . A map  $F : \mathcal{X} \rightarrow X$  is said to be a *universal mean* on  $X$  if it satisfies the conditions of anonymity-neutrality, unanimity, compatibility and monotonicity (with respect to the total order  $\preceq$  on  $X$ ).

**Remark 3.13.** We may wonder why in Definition 3.12 we have used the condition of monotonicity instead of the strict monotonicity that is inherent to Kolmogorov's Theorem 3.10. The reason is that strict monotonicity is an enormously restrictive condition. Notice, for instance, that given two elements  $x \neq y \in X$  such that  $x \preceq y$ , if strict monotonicity is imposed we should have a priori an infinite and order dense-in-itself<sup>4</sup> set of elements in  $X$  located between  $x$  and  $y$  as regards the linear order  $\preceq$ . To see this we may just notice that, for any  $n \in \mathbb{N}$ ,  $x = F(x, \dots (n \text{ times}) \dots, x) \preceq F(x, \dots (n-1 \text{ times}) \dots, x, y) \preceq F(x, \dots (n-2 \text{ times}) \dots, x, y, y) \preceq \dots \preceq F(y, \dots (n \text{ times}) \dots, y) = y$ , but all the intermediate elements should be pairwise different because of strict monotonicity. In particular, the set  $X$  must a fortiori be infinite. In spite of the real line  $\mathbb{R}$ , endowed with its usual linear order, allowing the existence of all these intermediate elements, we may not expect this to happen on a general total ordered set  $(X, \preceq)$ . An evident example of this situation appears when  $X$  is finite.

**Definition 3.14.** Given a nonempty set  $X$ , a *total preorder*  $\preceq$  on  $X$  is a binary relation that is transitive and total. The binary relation  $\sim$  defined by  $x \sim y \Leftrightarrow x \preceq y \preceq x$  ( $x, y \in X$ ) is said to be the *symmetric part* (also known as the *indifference*) associated to  $\preceq$ .

Now we furnish a characterization of the existence of a universal mean on a totally ordered set  $(X, \preceq)$  in terms of a suitable extension of the total order  $\preceq$  to a total preorder  $\preceq$  defined on  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$  and satisfying some properties ad hoc.

**Theorem 3.15.** Let  $(X, \preceq)$  be a totally ordered set. Let  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$ . Then there exists a universal mean  $F$  on  $X$  if and only if the total order  $\preceq$  admits an extension to a total preorder  $\preceq$  defined on  $\mathcal{X}$  and satisfying the following properties:

- (i) For every  $x, y \in X$  it holds true that  $x \preceq y \Leftrightarrow x \preceq y$ .
- (ii) For every  $n \in \mathbb{N}$  and  $(x_1, \dots, x_n) \in X^n$ , there exists a unique  $x \in X$  such that  $(x_1, \dots, x_n) \sim x$ .
- (iii) For any  $n \in \mathbb{N}$ , and  $(x_1, \dots, x_n) \in X^n$  it holds true that  $(x_1, \dots, x_n) \sim (x_{\sigma(1)}, \dots, x_{\sigma(n)})$  for any rearrangement  $\sigma$  of the set  $\{1, \dots, n\}$ .
- (iv) For every  $x \in X$ , and  $n \in \mathbb{N}$ , it holds true that  $(x, \dots (n \text{ times}) \dots, x) \sim x$ .

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<sup>4</sup>A totally ordered set  $(X, \preceq)$  is said to be *order dense-in-itself* if given  $x \neq y \in X$  with  $x \preceq y$ , there exists  $z \in X$  such that  $x \neq z, y \neq z$  and  $x \preceq z \preceq y$  holds true.

- (v) For any  $n, k \in \mathbb{N}$ ,  $x \in X$ , and  $(x_1, \dots, x_{n+k}) \in X^{n+k}$  it holds true that  $(x_1, \dots, x_n) \sim x \Rightarrow (x, \dots (n \text{ times}) \dots, x, x_{n+1}, \dots, x_{n+k}) \sim (x_1, \dots, x_{n+k})$ .
- (vi) For any  $n \in \mathbb{N}$ , and  $(x_1, \dots, x_n), (y_1, \dots, y_n) \in X^n$  it holds true that  $x_i \preceq y_i \ (1 \leq i \leq n) \Rightarrow (x_1, \dots, x_n) \precsim (y_1, \dots, y_n)$ .

*Proof.* To prove the direct implication, let  $F$  be a universal mean on  $X$ . Denote by  $F_n$  the restriction of  $F$  to  $X^n$  ( $n \in \mathbb{N}$ ). Given  $n, k \in \mathbb{N}$ ,  $(x_1, \dots, x_n) \in X^n$  and  $(y_1, \dots, y_k) \in X^k$ , we declare that  $(x_1, \dots, x_n) \precsim (y_1, \dots, y_k) \Leftrightarrow F(x_1, \dots, x_n) \preceq F(y_1, \dots, y_k)$ . It is straightforward to check that  $\precsim$  is a total preorder on  $X$  that accomplishes the properties (i)-(vi) of the statement.

To prove the converse implication, we define a map  $F : \mathcal{X} \rightarrow X$  as follows: Given  $n \in \mathbb{N}$  and  $(x_1, \dots, x_n) \in X^n$ , we take the unique element  $x \in X$  such that  $(x_1, \dots, x_n) \sim x$  and declare that  $F(x_1, \dots, x_n) = x$ . With this definition, it is now routine to see that  $F$  is a universal mean on  $X$ .  $\square$

### Examples 3.16.

- (i) Let  $X = \{a, b, c\}$ , with the total order  $\preceq$  given by  $a \preceq b \preceq c$ . We may extend this linear order  $\preceq$  to a total preorder  $\precsim$  defined on  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$  by declaring that for any  $n \in \mathbb{N}$ ,  $x \in X$ , and  $(x_1, \dots, x_n) \in X^n$  it holds true that  $(x, \dots (n \text{ times}) \dots, x) \sim x$  and  $(x_1, \dots, x_n) \sim b$  if  $b \in \{x_1, \dots, x_n\}$  or  $a, c \in \{x_1, \dots, x_n\}$ . It is straightforward to see that  $\precsim$  satisfies the conditions (i)-(vi) of the statement of Theorem 3.15. In addition, the corresponding mean  $F$  is associative. However, it fails to be a selection because  $F(a, c) = b$ . Finally, notice also that neither  $F$  is given by “taking maxima” nor by “taking minima” (see Remark 3.26 (ii) below).
- (ii) Once more, let  $X = \{a, b, c\}$ , with the total order  $\preceq$  given by  $a \preceq b \preceq c$ . We may extend this linear order  $\preceq$  to a total preorder  $\precsim$  defined on  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$  by declaring that for any  $n \in \mathbb{N}$ ,  $x \in X$ , and  $(x_1, \dots, x_n) \in X^n$  it holds true that  $(x, \dots (n \text{ times}) \dots, x) \sim x$ ,  $(x_1, \dots, x_n) \sim a$  if  $a \in \{x_1, \dots, x_n\}$ , and  $(x_1, \dots, x_n) \sim c$  if  $c \in \{x_1, \dots, x_n\}, a \notin \{x_1, \dots, x_n\}$ . Again, it is straightforward to see that  $\precsim$  satisfies the conditions (i)-(vi) in Theorem 3.15. Moreover, the corresponding mean  $F$  is an associative selection. But  $F$  is not given by taking maxima or else by taking minima, since  $F(a, b) = a$  whereas  $F(b, c) = c$ .

**Proposition 3.17.** *Let  $X$  be a nonempty set endowed with a total order  $\preceq$ . Let  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$ . Let  $F : \mathcal{X} \rightarrow X$  be a universal mean on  $X$ . Denote by  $F_n$  the restriction of  $F$  to  $X^n$  ( $n \in \mathbb{N}$ ). If  $F$  is a selection, then it is impossible to find  $a, b, c \in X$  such that  $a \neq b, a \neq c, b \neq c$ ,  $a \preceq b \preceq c$ , and  $F_2(a, b) = b = F_2(b, c)$ .*

*Proof.* If there exist  $a, b, c \in X$  such that  $a \neq b, a \neq c, b \neq c$  as well as  $a \preceq b \preceq c$  and  $F_2(a, b) = b = F_2(b, c)$ , then  $F_2(a, c)$  must be  $a$  or  $c$  because  $F$  is a selection. However, if  $F_2(a, c) = a$  we could not have that  $F_2(a, b) = b$  since  $F$  is monotone by hypothesis. So  $F_2(a, c) \neq a$ . But, if  $F_2(a, c) = c$ , again by monotonicity of  $F$  we could not have that  $F_2(b, c) = b$ . Thus we arrive to a contradiction.  $\square$

Last Examples 3.16 show that on a totally ordered set  $(X, \preceq)$  an associative selection that is a universal mean might still not coincide with the mean given by taking maxima nor with the one given by taking minima. To characterize these two special means, we introduce next definition.

**Definition 3.18.** Let  $(X, \preceq)$  be a totally ordered set. Let  $F : X \times X \rightarrow X$  be a selection (i.e:  $F(x, y) \in \{x, y\}$  holds true for every  $x, y \in X$ ). Then  $F$  is said to *agree with the total order*  $\preceq$  if either  $F(x, y) = x$  for every  $x, y \in X$  with  $x \preceq y$  or  $F(x, y) = y$  for every  $x, y \in X$  with  $x \preceq y$ .

**Remark 3.19.** In Examples 3.16 (ii) we show a universal mean  $F$  on a totally ordered set  $(X, \preceq)$  such that  $F$  is a selection, but  $F_2$  does not agree with  $\preceq$  since  $F(a, b) = a, F(b, c) = c$  and  $a \neq b, a \neq c, b \neq c$ , whereas  $a \preceq b \preceq c$ .

**Theorem 3.20.** Let  $X$  be a nonempty set endowed with a total order  $\preceq$ . Let  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$ . Let  $F : \mathcal{X} \rightarrow X$  be a universal mean on  $X$ . Denote by  $F_n$  the restriction of  $F$  to  $X^n$  ( $n \in \mathbb{N}$ ). The following statements are equivalent:

- (i)  $F$  is a selection such that  $F_2$  agrees with the total order  $\preceq$ .
- (ii) Either  $F_n(x_1, \dots, x_n) = \min\{x_1, \dots, x_n\}$  holds true for every  $n \in \mathbb{N}$  and  $(x_1, \dots, x_n) \in X^n$  or else, alternatively,  $F_n(x_1, \dots, x_n) = \max\{x_1, \dots, x_n\}$  holds for all  $n \in \mathbb{N}$  and  $(x_1, \dots, x_n) \in X^n$ . (In particular, the universal mean  $F$  is associative).

*Proof.* Since the converse implication follows directly from definitions, we only prove the direct implication (i)  $\Rightarrow$  (ii). To do so, we follow an iterative process. Let  $n \in \mathbb{N}$  and  $(x_1, \dots, x_n) \in X^n$ . Assume without loss of generality that  $x_1 \preceq \dots \preceq x_n$ . Let  $x_k$  be the first element (if any) that is different from  $x_1$  in  $\{x_1, \dots, x_n\}$ . (If all the elements were equal, then obviously  $F_n(x_1, \dots, x_n) = x_1$ ). Suppose that  $F_2(x_1, x_k) = x_1$ . Let  $x_j$  be the first element (if any) that is different from both  $x_1$  and  $x_k$ . Then  $F_2(x_k, x_j) = x_k$ , because  $F$  is a selection compatible with  $\preceq$ . (Here, if  $x_k = x_{k+1} = \dots = x_n$ , then obviously  $F_2(x_k, x_l) = x_k$  if  $k < l$ ). Thus using the property of compatibility of the universal mean  $F$ , it follows that  $F_3(x_1, x_k, x_j) = F_3(x_1, x_k, x_k) = F_3(x_1, x_1, x_k) = F_3(x_1, x_1, x_1) = x_1$ . Following an inductive process, we obtain that  $F_n(x_1, \dots, x_n) = x_1 = \min\{x_1, \dots, x_n\}$ . In an entirely analogous way, if  $F_2(x_1, x_k) = x_k$  we would finally arrive at  $F_n(x_1, \dots, x_n) = x_n = \max\{x_1, \dots, x_n\}$ . (Alternatively, we may also use Proposition 3.17 to easily prove this last claim).

To conclude the proof, suppose that there exist  $n, m \in \mathbb{N}, (x_1, \dots, x_n) \in X^n, (y_1, \dots, y_m) \in X^m$  such that  $x_1 \preceq \dots \preceq x_n$  and also  $y_1 \preceq \dots \preceq y_m$ . Suppose also that  $x_1 \neq x_n$  and  $y_1 \neq y_m$ . Finally assume, by way of contradiction, that  $F_n(x_1, \dots, x_n) = x_1$  but  $F_m(y_1, \dots, y_m) = y_m$ . Notice that  $F_2(x_1, x_n) = x_1$  by compatibility of  $F$ , and the agreement of  $F_2$  with  $\preceq$ . Similarly  $F_2(y_1, y_m) = y_m$ . But this contradicts the fact of  $F_2$  agreeing with the linear order  $\preceq$ . □

Next battery of Definitions 3.21 to 3.23 and Theorem 3.24 will be used to prove that in some particular cases of total orders, universal means can always be defined following the steps of Remark 3.11 (iv) above.

**Definition 3.21.** Let  $X$  be a nonempty set endowed with a total order  $\preceq$ . The total order  $\preceq$  is said to be *representable* if there exists a function  $u : X \rightarrow \mathbb{R}$ , known as a *utility function*, such that  $x \preceq y \Leftrightarrow u(x) \leq u(y)$  holds true for every  $x, y \in X$ . The *asymmetric part* of  $\preceq$ , that we denote  $\prec$ , is defined by  $a \prec b \Leftrightarrow a \preceq b; a \neq b$  ( $a, b \in X$ ). The total order  $\preceq$  is said to be *perfectly separable* if there exists a countable subset  $D \subseteq X$  such that for every  $x, y \in X$  with  $x \prec y$ , there exists an element  $d \in D$  such that  $x \preceq d \preceq y$  holds true.

**Definition 3.22.** Let  $X$  be a nonempty set endowed with a total order  $\preceq$ . Given  $x \in X$ , the sets  $L_x = \{z \in X : z \prec x\}$  and  $R_x = \{t \in X : x \prec t\}$  are respectively said to be the left and right *contour set* of  $x$  as regards  $\preceq$ . A pair of elements  $a, b \in X$  such that  $a \prec b$  and  $R_a \cap L_b = \emptyset$  is said to define a *gap*. The topology  $\tau_{\preceq}$  on  $X$ , a subbasis of which is given by the family  $\{\emptyset\} \cup \{X\} \cup \{L_x : x \in X\} \cup \{R_y : y \in X\}$ , is called the *order topology* on  $X$ .

**Definition 3.23.** Let  $X$  be a nonempty set endowed with a total order  $\preceq$ . A nonempty subset  $Y \subseteq X$  is said to be *bounded by above* with respect to  $\preceq$  if there exists some element  $a \in X$  such that  $y \preceq a$  holds for every  $y \in Y$ . The element  $a$  is said to be an *upper bound* of  $a$ . If a smallest upper bound of  $y$  exists, it is said to be the *supremum* of  $Y$ . Finally,  $X$  is said to be *Dedekind-complete* if every subset  $Y \subseteq X$  that is bounded by above has a supremum.

**Theorem 3.24.** Let  $X$  be a nonempty set endowed with a total order  $\preceq$ . The following statements are equivalent:

- (i) The total order  $\preceq$  is perfectly separable.
- (ii) The total order  $\preceq$  is representable.
- (iii) The total order  $\preceq$  is representable through a real-valued utility function  $U : (X, \tau_{\preceq}) \rightarrow (\mathbb{R}, \tau_u)$  which is continuous with respect to the order topology  $\tau_{\preceq}$  on  $X$  and the usual topology  $\tau_u$  on the real line.

In addition,  $(X, \tau_{\preceq})$  is homeomorphic to an interval of the real line endowed with the usual topology if and only if  $\preceq$  is perfectly separable, Dedekind-complete and without gaps.

*Proof.* These results are classic in the theory of numerical representations of ordered structures. See e.g. [25], the first three chapters in [11], p.3 in [34] and p. 52 in [38] for several proofs and further details.  $\square$

**Corollary 3.25.** Let  $X$  be a nonempty set endowed with a total order  $\preceq$  that is perfectly separable, Dedekind-complete and without gaps. Then there exists a universal mean on  $X$ .

*Proof.* By Theorem 3.24, the topological space  $(X, \tau_{\preceq})$  is homeomorphic to an interval  $J$  of the real line endowed with the usual topology  $\tau_u$ . On the interval  $J$  we can consider a regular mean in the sense of Remark 3.11 (iii). (For instance, the classical arithmetic mean suffices). Then a universal mean can be obtained on  $X$  following the ideas of Remark 3.11 (iv).  $\square$

**Remarks 3.26.**

- (i) Coming back to the search for universal means on a *general* totally ordered set  $(X, \preceq)$ , we cannot expect things to be as easy and direct as in the previous corollary, even in case of representability through a utility function  $u$ . Despite identifying each  $x \in X$  to the real number  $u(x)$ , it may happen that, given  $x_1, \dots, x_n \in X^n$ , a typical “mean” of the real numbers  $u(x_1), \dots, u(x_n) \in \mathbb{R}$  as, for instance, its arithmetic mean  $\frac{u(x_1) + \dots + u(x_n)}{n}$ , does not belong to  $u(X)$ , so that we are impeded to identify it to an element of the set  $X$ . An appealing particular case appears when the set  $X$  is finite.
- (ii) Fortunately, *universal means always exist on a totally ordered set  $(X, \preceq)$* . Perhaps the most evident examples are the selections of minima and maxima. Thus, for any  $n \in \mathbb{N}$  and  $x_1, \dots, x_n \in X^n$ , we define  $m_n(x_1, \dots, x_n) = \min\{x_1, \dots, x_n\}$ , and  $M_n(x_1, \dots, x_n) = \max\{x_1, \dots, x_n\}$ . The map  $m$  (respectively, the map  $M$ ), from  $\mathcal{X}$  into  $X$ , whose restriction to  $X^n$  is  $m_n$  (respectively,  $M_n$ ) is actually a universal mean on  $X$ . By the way, both of them are selections.
- (iii) The existence of a universal mean  $F = (F_n)_{n=1}^\infty$  on a representable totally ordered set  $(X, \preceq)$  extends the total order  $\preceq$  on  $X$  to a total preorder on  $\mathcal{X}$ , as stated in Theorem 3.15. But now, since  $\preceq$  is representable, an alternative way to extend  $\preceq$  is the following: if  $u$  is a utility function that represents  $X$ , then the binary relation  $\preceq$  on  $\mathcal{X}$  defined by  $(x_1, \dots, x_m) \preceq (y_1, \dots, y_n) \Leftrightarrow u(F_m(x_1, \dots, x_m)) \leq u(F_n(y_1, \dots, y_n))$ , for all  $m, n \in \mathbb{N}$  and  $(x_1, \dots, x_m, y_1, \dots, y_n) \in X^{m+n}$ , is indeed a total preorder on  $\mathcal{X}$ , and it is clear that  $\preceq$  extends  $\preceq$ . In particular, this idea can straightforwardly be also used to extend the linear order  $\preceq$  in  $X$  to a total preorder  $\preceq^*$  on the power set of  $X$ . The study of extensions of rankings from a set to its power set accomplishing some criteria previously established is typical in Mathematical Economics and Social Choice. (See e.g. [7].)

**3.4. Finite means on lattices**

Generalizing a further step the ideas analyzed in the previous subsection for total orders, now we discuss the definition of suitable means on lattices.

**Definition 3.27.** Let  $(L, \vee, \wedge)$  be a nonempty lattice, whose associated partial order is  $\preceq$ . Let  $\mathcal{L} = \bigcup_{n=1}^\infty L^n$ . A function  $F : \mathcal{L} \rightarrow L$  is said to be a *lattice mean* on  $L$  if it satisfies the conditions of anonymity-neutrality, unanimity, compatibility and, in addition, its restriction to each  $L^n$  is monotone in each variable

with respect to the partial order  $\preceq$  that generates the latticial operations  $\vee$  and  $\wedge$ .

First we may notice that, as in Remark 3.26 (ii), we have at least two suitable means on any lattice.

**Remark 3.28.** Observe that if  $(L, \vee, \wedge)$  be a nonempty lattice whose associated partial order is  $\preceq$ , and  $\mathcal{L} = \bigcup_{n=1}^{\infty} L^n$ , then the function  $\bigvee : \mathcal{L} \rightarrow L$  (respectively and dually, the function  $\bigwedge : \mathcal{L} \rightarrow L$ ), given by  $\bigvee(l_1, \dots, l_n) = l_1 \vee \dots \vee l_n$  (respective and dually, given by  $\bigwedge(l_1, \dots, l_n) = l_1 \wedge \dots \wedge l_n$ ) for any  $n \in \mathbb{N}$  and  $(l_1, \dots, l_n) \in L^n$ , is a latticial mean on  $L$ .

At this stage, it is important to pay attention to the following key fact: Unlike the case of a total ordered set, now latticial means, defined through the operations  $\vee$  (meet) or  $\wedge$  (join) on a nonempty lattice  $(L, \vee, \wedge)$ , may fail to be selections. Notice that given  $n \in \mathbb{N}$  and  $(l_1, \dots, l_n) \in L^n$ , it may happen that  $\bigvee(l_1, \dots, l_n)$  or  $\bigwedge(l_1, \dots, l_n)$  could be different from any element in  $\{l_1, \dots, l_n\}$ .

### 3.5. Finite means on semigroups

A lattice  $(L, \vee, \wedge)$  can, in particular, be considered as an *algebraic structure*, in which the nonempty set  $L$  is endowed with two binary operations, namely the join  $\vee$  and the meet  $\wedge$ . Both of them are *associative*, and, of course, they are related between them (e.g: each of them is distributive with respect to the other one). From an algebraic point of view, we may pass to consider weaker structures, in which a nonempty set  $X$  is endowed with a binary operation  $*$  that satisfies some structural properties. We will focus here on the structure of a *semigroup*.

**Definition 3.29.** Let  $(S, *)$  be a semigroup. If there exists an element  $e \in S$  such that  $e * s = s * e = s$  holds for every  $s \in S$ , then the semigroup is said to be a *monoid*, and the element  $e$  (which is unique) is called the *neutral* (or “identity”) element of the monoid. If the operation  $*$  is commutative, then the semigroup is also said to be *commutative* (or “Abelian”).

Given a semigroup  $(S, *)$  we had already given a definition of a *semigroup  $n$ -mean* in the previous section (see Definition 2.8 above). But each  $n$ -mean is only defined for a fixed  $n$ . So, we pass to consider  $\mathcal{S} = \bigcup_{n=1}^{\infty} S^n$ , and define the concept of a *semigroup (global) mean*, as follows.

**Definition 3.30.** Let  $(S, *)$  be a semigroup. Consider a map  $F : \mathcal{S} \rightarrow S$ . Let  $F_n$  stand for the restriction of  $F$  to  $S^n$  ( $n \in \mathbb{N}$ ). The function  $F$  is called a *semigroup mean on  $S$*  if it satisfies the conditions of anonymity-neutrality and unanimity, and, in addition, for every  $n \in \mathbb{N}$ , and  $(s_1, \dots, s_n) \in S^n$ , it holds true that  $s_1 * \dots * s_n = s * \dots (n \text{ times}) * s$ , with  $s = F_n(s_1, \dots, s_n)$ .

### Examples 3.31.

- (i) The arithmetic mean on the additive real line  $(\mathbb{R}, +)$  is a semigroup mean on the structure  $(\mathbb{R}, +)$ , which is, in particular, a semigroup.



- (ii) If  $(L, \vee, \wedge)$  is a lattice, both the structures  $(L, \vee)$  and  $(L, \wedge)$  are semigroups. Moreover, the meet latticial mean (respectively, the join latticial mean) is a semigroup mean as regards  $(L, \vee)$  (respectively, as regards  $(L, \wedge)$ ).

**Proposition 3.32.** *Let  $(S, *)$  be a semigroup. If there exists a semigroup mean  $F$  on  $S$ , then the semigroup  $(S, *)$  is commutative.*

*Proof.* Let  $s, t \in S$ . Let  $x \in S$  be such that  $F(s, t) = x = F(t, s)$ . Notice that, by definition of a semigroup mean, we have that  $s * t = x * x = t * s$ .  $\square$

The definition of a semigroup mean inspires a new concept of a *general associative mean* on nonempty sets, without any structure given a priori.

**Definition 3.33.** Let  $X$  be a nonempty set. Let  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$ . A map  $F : \mathcal{X} \rightarrow X$  is said to be a *general associative mean* on  $X$  if it satisfies the conditions of anonymity-neutrality, unanimity, compatibility and associativity.

**Remark 3.34.** Let  $X$  be a nonempty set and  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$ . Let  $F : \mathcal{X} \rightarrow X$  be a general associative mean on  $X$ . Denote by  $F_n$  the restriction of  $F$  to  $X^n$  ( $n \in \mathbb{N}$ ). Then  $X$  becomes a commutative semigroup when equipped with the binary operation  $*$  given by  $x * y = F_2(x, y) = F_2(y, x)$  ( $x, y \in X$ ).

**Definition 3.35.** Let  $X$  be a nonempty set and  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$ . Let  $F : \mathcal{X} \rightarrow X$  be a map that satisfies the conditions of anonymity-neutrality, unanimity and compatibility. Then  $F$  is said to be *iterative* if for every  $n \geq 2 \in \mathbb{N}$  and every  $(x_1, \dots, x_n) \in X^n$ , it holds true that  $F_n(x_1, \dots, x_n) = F_2(F_{n-1}(x_1, \dots, x_{n-1}), x_n)$ .

**Theorem 3.36.** *Let  $X$  be a nonempty set and  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$ . Let  $F : \mathcal{X} \rightarrow X$  be an iterative general associative mean on  $X$ . Denote by  $F_2$  the restriction of  $F$  to  $X^2$ , and endow  $X$  with the binary operation  $*$  defined by  $x * y = F_2(x, y)$  ( $x, y \in X$ ). Then  $F$  is actually a semigroup mean on the semigroup structure  $(X, *)$ .*

*Proof.* We only need to check that for every  $n \in \mathbb{N}$ , and  $(x_1, \dots, x_n) \in X^n$ , it holds true that  $x_1 * \dots * x_n = s * \dots (n \text{ times}) * s$ , with  $s = F_n(s_1, \dots, s_n)$ . To see this, notice that by iterativity we inductively obtain that  $F_n(x_1, \dots, x_n) = x_1 * \dots * x_n$ . Let  $s = F_n(x_1, \dots, x_n)$ . By compatibility of  $F$  we obtain  $F_n(x_1, \dots, x_n) = F_n(s, \dots (n \text{ times}) \dots, s)$ . And, again by iterativity, we conclude that  $F_n(s, \dots (n \text{ times}) \dots, s) = s * \dots (n \text{ times}) * s$ .  $\square$

### 3.6. Other miscellaneous kinds of finite means

In Section 2 we already mentioned some definitions of  $n$ -means in various contexts. Now we may directly pass to consider the corresponding means as functions  $M$  defined on  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$  and taking values on  $X$ .

**Definition 3.37.** Let  $(X, \tau)$  be a topological space and  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$ . A map  $F : \mathcal{X} \rightarrow X$  is said to be a *topological mean* if it satisfies the conditions of anonymity-neutrality and unanimity, and, in addition, the restriction  $F_n$  of  $F$  to  $X^n$  is a continuous map for every  $n \in \mathbb{N}$ , as regards the topology  $\tau$  on  $X$  and the corresponding product topology on  $X^n$ .

**Definition 3.38.** Let  $X$  be a nonempty set endowed with a binary operation  $\circ$ . Let  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$ . A map  $F : \mathcal{X} \rightarrow X$  is said to be an *algebraic mean* if it satisfies the conditions of anonymity-neutrality and unanimity, and, in addition, for every  $n \in \mathbb{N}$ , the restriction  $F_n$  of  $F$  to  $X^n$  accomplishes that  $F(x_1 \circ y_1, \dots, x_n \circ y_n) = F(x_1, \dots, x_n) \circ F(y_1, \dots, y_n)$ , for every  $(x_1, \dots, x_n), (y_1, \dots, y_n) \in X^n$ .

A method based on a recursive application of a bivariate mean in a binary tree construction has been explored in [9] and [27] for extending that bivariate mean to  $n$ -variated weighted means.

### 3.7. General means on a nonempty set

To conclude this Section 3 we may try to choose a definition of a suitable *mean* for the general case of a nonempty set  $X$  with no additional structure given a priori.

**Definition 3.39.** Let  $X$  be a nonempty set and  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$ . A map  $F : \mathcal{X} \rightarrow X$  is said to be a *general mean* if it satisfies the conditions of anonymity-neutrality, unanimity and compatibility.

The reasons that support this last definition are the following: On the one hand, the conditions imposed to a general mean in Definition 3.39 appear in the definitions of regular means in the real line, universal means on totally ordered sets, and latticial means. On the other hand, a general mean is also a semigroup mean if we endow  $S$  with a “trivial” operation  $*$  defined as follows: choose an element  $e \in S$  and declare that  $x * e = e = e * x$  holds true for every  $x \in X$ . In addition, with that operation  $*$ , a general mean is also an algebraic mean in the sense of Definition 3.37. Furthermore, if  $X$  is endowed with the discrete topology, a general mean also becomes a topological mean, again in the sense of Definition 3.37. Finally, in order to give a definition as abstract as possible, we should keep to a minimum the restrictions imposed a priori. Anonymity-neutrality, unanimity and compatibility provide a clear account of what we should require to any finite mean, independently from the context.

## 4. Iterativity and related properties

Despite having considered in Section 3 several kinds of a mean on a nonempty set  $X$  as maps  $F$  from  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$  taking values in  $X$  and satisfying suitable properties, a priori, the restrictions  $F_n$  and  $F_m$  of  $F$  to  $X^n$  and  $X^m$  might be unrelated if  $n \neq m$ . Perhaps only the condition of compatibility, when available for  $F$ , establishes a first relationship between different restrictions of

$F$  to Cartesian products of several copies of  $X$ . Nevertheless, no way to obtain an  $F_n$  from other  $F_k$  with  $k \neq n$  is given at first hand. Already in Section 2 we gave some ideas about how to obtain  $F_n$  from  $F_2$ , when possible. Also, in Definition 3.35 we introduced a concept of iterativity for means. In this Section 4 we pay a deeper attention to this key concept.

Matching Definitions 3.35 and 3.39, a *general iterative mean* on a nonempty set  $X$ , is a map  $F$  from  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$  into  $X$  that satisfies the conditions of anonymity-neutrality, unanimity, compatibility and iterativity. Now we introduce new conditions that are equivalent to iterativity of a general mean defined on a nonempty set  $X$ , so giving rise to alternative definitions or versions of that key property.

**Definition 4.1.** Let  $X$  be a nonempty set and  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$ . Let  $F : \mathcal{X} \rightarrow X$  be a general mean on  $X$ . Denote by  $F_n$  the restriction of  $F$  to  $X^n$  ( $n \in \mathbb{N}$ ). The map  $F$  is said to be:

- (i) *reducible* if for every  $a \in X$ ,  $n, k \in \mathbb{N}$  and  $(x_1, \dots, x_k) \in X^k$  it holds true that  $F_{n+k}(a, \dots, (n \text{ times}) \dots, a, x_1, \dots, x_k) = F_{k+1}(a, x_1, \dots, x_k)$ ,
- (ii) *shrinking* if for every  $n \in \mathbb{N}$  and  $a, b \in X$  it holds true that  $F_{n+1}(a, \dots, (n \text{ times}) \dots, a, b) = F_2(a, b)$ ,
- (iii) *compressible* if for every  $k, n \in \mathbb{N}$  and  $(x_1, \dots, x_{n+k}) \in X^{n+k}$  it holds true that  $F_{n+k}(x_1, \dots, x_{n+k}) = F_{k+1}(F_n(x_1, \dots, x_n), x_{n+1}, \dots, x_{n+k})$ .

**Remark 4.2.** Notice that the definition of compressibility extends that of iterativity, that corresponds to the case  $k = 1$  in the definition of compressibility. (See Definition 3.35).

**Theorem 4.3.** Let  $F$  be a general mean on a nonempty set  $X$ . The following statements are equivalent:

- (i)  $F$  is reducible,
- (ii)  $F$  is shrinking,
- (iii)  $F$  is iterative,
- (iv)  $F$  is associative,
- (v)  $F$  is compressible.

*Proof.* We will follow the schema: (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (i)  $\Rightarrow$  (v)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (ii).

The implication (i)  $\Rightarrow$  (ii) is trivial. Just take  $k = 1$ .

Let us prove now that (ii)  $\Rightarrow$  (iii): Assume that  $F$  is shrinking. Let  $n \in \mathbb{N}$  and  $(x_1, \dots, x_n) \in X^n$ . By compatibility, it follows that  $F_n(x_1, \dots, x_n) = F_n(F_{n-1}(x_1, \dots, x_{n-1}), \dots, (n-1 \text{ times}) \dots, F_{n-1}(x_1, \dots, x_{n-1}), x_n)$ . Since  $F$  is

shrinking, we get  $F_n(F_{n-1}(x_1, \dots, x_{n-1}), \dots (n-1 \text{ times}) \dots, F_{n-1}(x_1, \dots, x_{n-1}), x_n) = F_2(F_{n-1}(x_1, \dots, x_{n-1}), x_n)$ . So  $F$  is iterative.

To prove the implication (iii)  $\Rightarrow$  (i), assume that  $F$  is iterative. We will prove that  $F$  is reducible by induction on  $k \in \mathbb{N}$ . If  $k = 1$ , given  $n \in \mathbb{N}$  and  $a, b \in X$ , by iterativity it follows that  $F_{n+k}(a, \dots (n \text{ times}) \dots, a, b) = F_2(F_n(a, \dots (n \text{ times}) \dots, a), b) = F_2(a, b)$ . Therefore  $F$  is, in particular, reducible. Assume now that for  $a \in X$ ,  $n, k \in \mathbb{N}$  and  $(x_1, \dots, x_k) \in X^k$  it holds true that  $F_{n+k}(a, \dots (n \text{ times}) \dots, a, x_1, \dots, x_k) = F_{k+1}(a, x_1, \dots, x_k)$ . To conclude, it is enough to prove now that for any  $x_{k+1} \in X$  it also holds that  $F_{n+k+1}(a, \dots (n \text{ times}) \dots, a, x_1, \dots, x_k, x_{k+1}) = F_{k+2}(a, x_1, \dots, x_k, x_{k+1})$ . To see this, notice that by iterativity we have that  $F_{n+k+1}(a, \dots (n \text{ times}) \dots, a, x_1, \dots, x_k, x_{k+1}) = F_2(F_{n+k}((a, \dots (n \text{ times}) \dots, a, x_1, \dots, x_k), x_{k+1}))$ . By the induction hypothesis,  $F_{n+k}((a, \dots (n \text{ times}) \dots, a, x_1, \dots, x_k) = F_{k+1}(a, x_1, \dots, x_k)$ , so that  $F_2(F_{n+k}((a, \dots (n \text{ times}) \dots, a, x_1, \dots, x_k), x_{k+1})) = F_2(F_{k+1}(a, x_1, \dots, x_k), x_{k+1})$ . Finally, by iterativity again,  $F_2(F_{k+1}(a, x_1, \dots, x_k), x_{k+1}) = F_{k+2}(a, x_1, \dots, x_k, x_{k+1})$ . Therefore  $F$  is reducible.

To see that (i)  $\Rightarrow$  (v), just take  $a = F_n(x_1, \dots, x_n)$  and notice that, by compatibility,  $F_{n+k}(x_1, \dots, x_{n+k}) = F_{n+k}(a, \dots (n \text{ times}) \dots, a, x_{n+1}, x_{n+k})$ . Then use that  $F$  is reducible.

To see that (v)  $\Rightarrow$  (iii), just take  $k = 1$  in the definition of compressibility, as already commented in Remark 4.2.

Let us prove now that (iii)  $\Rightarrow$  (iv): Given  $a, b, c \in X$  we have that  $F_3(a, b, c) = F_2(F_2(a, b), c)$  by iterativity. Also  $F_3(a, b, c) = F_3(b, c, a)$  by anonymity-neutrality. By iterativity again we get  $F_3(b, c, a) = F_2(F_2(b, c), a)$ . Finally, once more by anonymity-neutrality,  $F_2(F_2(b, c), a) = F_2(a, F_2(b, c))$ . We conclude that  $F_2(F_2(a, b), c) = F_2(a, F_2(b, c))$ . So  $F$  is associative.

Finally, let us see that (iv)  $\Rightarrow$  (ii): Let  $a, b, c \in X$ . Having Theorem 3.3 in mind, we have that  $F_3(a, b, c) = F_{12}(a, a, a, a, b, b, b, b, c, c, c, c) = F_{12}(a, b, c, a, b, a, c, b, c, a, b, c) = F_{12}(a, F_2(b, c), F_2(b, c), a, b, F_2(a, c), F_2(a, c), b, c, F_2(a, b), F_2(a, b), c) = F_{12}(F_2(a, F_2(b, c)), F_2(a, F_2(b, c)), F_2(a, F_2(b, c)), F_2(a, F_2(b, c)), F_2(b, F_2(a, c)), F_2(b, F_2(a, c)), F_2(b, F_2(a, c)), F_2(b, F_2(a, c)), F_2(c, F_2(a, b)), F_2(c, F_2(a, b)), F_2(c, F_2(a, b)), F_2(c, F_2(a, b)))$ . By associativity and anonymity-neutrality, this yields  $F_{12}(F_2(a, F_2(b, c)), \dots (twelve \text{ times}) \dots, F_2(a, F_2(b, c))) = F_2(a, F_2(b, c))$  by unanimity. Hence  $F_3(a, b, c) = F_2(a, F_2(b, c))$ . So  $F_3(a, b, c) = F_2(F_2(a, b), c)$  by associativity. Using this we can inductively prove that  $F$  is shrinking. Notice that for  $n = 3$ ,  $F_3(a, a, b) = F_2(F_2(a, a), b) = F_2(a, b)$  by unanimity. For a given  $n$  we have that  $F_{n+1}(a, \dots (n \text{ times}) \dots, a, b) = F_{n+1}(a, \dots (n-1 \text{ times}) \dots, a, F_2(a, b), F_2(a, b)) = F_{n+1}(a, \dots (n-2 \text{ times}) \dots, a, F_2(a, F_2(a, b)), F_2(a, F_2(a, b)), F_2(a, b)) = F_{n+1}(a, \dots (n-2 \text{ times}) \dots, a, F_3(a, a, b), F_3(a, a, b), F_2(a, b)) = F_{n+1}(a, \dots (n-2 \text{ times}) \dots, a, F_2(a, b), F_2(a, b), F_2(a, b))$ . Proceeding in this way we easily arrive to  $F_{n+1}(a, \dots (n \text{ times}) \dots, a,$

$b) = F_{n+1}(F_2(a, b), \dots (n+1 \text{ times}) \dots, F_2(a, b)) = F_2(a, b)$  by unanimity.

This concludes the proof.  $\square$

**Example 4.4.** As already commented in Remark 3.13, the condition of strict monotonicity that appears in Kolmogorov's approach in the definition of the concept of a regular mean in the real line is too restrictive. Another key fact that comes from strict monotonicity of regular means is that none of them is associative. Let us see why: If  $F$  is a regular mean on the real line, and  $F_2$  is the restriction of  $F$  to the real plane  $\mathbb{R}^2$ , we observe that  $1 = F_2(1, 1) < F_2(1, 2) < F_2(2, 2) = 2 < F_2(2, 3) < F_2(3, 3) = 3$  holds true by unanimity and strict monotonicity. Moreover  $F_2(1, F_2(2, 3)) < F_2(F_2(1, 2), 3)$  by strict monotonicity, since  $1 < F_2(1, 2)$  and also  $F_2(2, 3) < 3$ . Therefore  $F_2(1, F_2(2, 3)) \neq F_2(F_2(1, 2), 3)$ , so that  $F$  fails to be associative.

Now we show some other (more restrictive) sufficient conditions for iterativity. To do so, we introduce the following definition.

**Definition 4.5.** Let  $X$  be a nonempty set and  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$ . Let  $F : \mathcal{X} \rightarrow X$  be a general mean on  $X$ . Let  $F_n$  stand for the restriction of  $F$  to  $X^n$  ( $n \in \mathbb{N}$ ). An element  $e \in X$  is said to be:

- (i) *reductive* as regards  $F$ , if  $F_n(x, \dots (n-1 \text{ times}) \dots, x, e) = x$  holds true for every  $x \in X$  and  $n \in \mathbb{N}$ ,
- (ii) *neutral* as regards  $F$ , if  $F_n(x, e, \dots (n-1 \text{ times}) \dots, e) = x$  holds true for every  $x \in X$  and  $n \in \mathbb{N}$ .

**Proposition 4.6.** Let  $X$  be a nonempty set and  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$ . Let  $F$  be a general mean on  $X$ . As regards  $F$ , an element  $e \in X$  is reductive if and only if it is neutral.

*Proof.* Suppose that  $e$  is reductive. Given  $n \in \mathbb{N}$  and  $x \in X$ , it follows that  $F_n(x, e, \dots (n-1 \text{ times}) \dots, e) = F_n(F_{n-1}(x, e, \dots (n-2 \text{ times}) \dots, e), \dots (n-1 \text{ times}) \dots, F_{n-1}(x, e, \dots (n-2 \text{ times}) \dots, e), e) = F_{n-1}(x, e, \dots (n-2 \text{ times}) \dots, e) = \dots = F_2(x, e) = x$ . Therefore  $e$  is neutral.

Conversely, let us assume that  $e$  is neutral. Given now  $n \in \mathbb{N}$  and  $x \in X$ , it follows that  $F_n(x, \dots (n-1 \text{ times}) \dots, x, e) = F_n(F_{n-1}(x, e, \dots (n-2 \text{ times}) \dots, e), \dots (n-1 \text{ times}) \dots, F_{n-1}(x, e, \dots (n-2 \text{ times}) \dots, e), e)$ . By generalized bisymmetry,  $F_n(F_{n-1}(x, e, \dots (n-2 \text{ times}) \dots, e), \dots (n-1 \text{ times}) \dots, F_{n-1}(x, e, \dots (n-2 \text{ times}) \dots, e), e) = F_n(F_{n-1}(x, \dots (n-1 \text{ times}) \dots, x), F_{n-1}(e, \dots (n-1 \text{ times}) \dots, e), \dots (n-2 \text{ times}) \dots, F_{n-1}(e, \dots (n-1 \text{ times}) \dots, e), e) = F_n(x, e, \dots (n-1 \text{ times}) \dots, e) = x$ . Hence  $e$  is reductive.  $\square$

**Theorem 4.7.** Let  $X$  be a nonempty set and  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$ . Let  $F : \mathcal{X} \rightarrow X$  be a general mean on  $X$ . If  $F$  has a reductive element  $e$ , then it is iterative.

*Proof.* Given  $n \in \mathbb{N}$  and  $(x_1, \dots, x_n) \in X^n$ , we get  $F_n(x_1, \dots, x_{n-1}, e) = F_n(F_{n-1}(x_1, \dots, x_{n-1}), \dots (n-1 \text{ times}) \dots, F_{n-1}(x_1, \dots, x_{n-1}), e) = F_{n-1}(x_1, \dots, x_{n-1})$ . Furthermore,  $F_n(x_1, \dots, x_n) = F_n(x_1, F_{n-1}(x_2, \dots (n-2 \text{ times}) \dots, x_2, e), \dots, F_{n-1}(x_n, \dots (n-2 \text{ times}) \dots, x_n, e)$ . By generalized bisymmetry,  $F_n(x_1, F_{n-1}(x_2, \dots (n-2 \text{ times}) \dots, x_2, e), \dots, F_{n-1}(x_n, \dots (n-2 \text{ times}) \dots, x_n, e)) = F_n(x_1, F_{n-1}(x_2, \dots x_n), \dots (n-2 \text{ times}) \dots, F_{n-1}(x_2, \dots, x_n), F_{n-1}(e, \dots (n-1 \text{ times}) \dots, e)) = F_n(x_1, F_{n-1}(x_2, \dots, x_n), \dots (n-2 \text{ times}) \dots, F_{n-1}(x_2, \dots, x_n), F_{n-1}(x_2, \dots, x_n), e) = F_{n-1}(x_1, F_{n-1}(x_2, \dots, x_n), \dots (n-2 \text{ times}) \dots, F_{n-1}(x_2, \dots, x_n))$ . Calling  $a = F_{n-1}(x_2, \dots, x_n)$ , we have that  $F_{n-1}(x_1, F_{n-1}(x_2, \dots, x_n), \dots, (n-2 \text{ times}) \dots F_{n-1}(x_2, \dots, x_n)) = F_{n-1}(x_1, a, \dots, (n-2 \text{ times}) \dots, a) = F_{n-2}(x_1, F_{n-2}(a, \dots, (n-2 \text{ times}) \dots, a), \dots, (n-3 \text{ times}) \dots, F_{n-2}(a, \dots, (n-2 \text{ times}) \dots, a)) = F_{n-2}(x_1, a, \dots, (n-3 \text{ times}) \dots, a)$ . So, proceeding in the same way, we finally arrive at  $F_n(x_1, \dots, x_n) = F_2(x_1, a) = F_2(x_1, F_{n-1}(x_2, \dots, x_n))$ . By anonymity-neutrality  $F_n(x_1, \dots, x_n) = F_n(x_n, x_1, \dots, x_{n-1})$ . Hence  $F_n(x_1, \dots, x_n) = F_2(x_n, F_{n-1}(x_1, \dots, x_{n-1})) = F_2(F_{n-1}(x_1, \dots, x_{n-1}), x_n)$ . Therefore  $F$  is iterative.  $\square$

**Remark 4.8.** The converse of Theorem 4.7 is not true. To see that, we consider again Example 3.16 (i). Notice that the general mean  $F$  built there has no neutral element. As a matter of fact, the element  $a$  cannot be neutral because  $F_2(c, a) = b \neq c$ . The element  $b$  can not be neutral, either, because  $F_2(a, b) = b \neq a$ . And the element  $c$  fails to be neutral because  $F_2(a, c) = b \neq a$ . However, it is straightforward to see that  $F$  is iterative by its own definition.

## 5. Miscellaneous applications

It is clear that the use of means in different contexts gives rise to a wide sort of applications. We outline here some of them, in the light of the results achieved in the previous sections.

Depending on the context, perhaps some classical concepts do not correspond exactly to the notion of a general mean on a nonempty set introduced before in Definition 3.39. Some conditions as, e.g., unanimity or anonymity-neutrality are sometimes dropped. This is made in order to study, for instance, situations of lack of symmetry or commutativity. However, many of the ideas and results introduced till this point can also be adapted somehow to these new settings.

### 5.1. Social Choice

Suppose that a finite set of individuals, say  $X = \{x_1, \dots, x_n\}$ , establish rankings on a finite set of objects, say  $Y$ . Each individual  $x_i \in X$  defines an ordering or ranking on  $Y$ . Let us assume that the ordering defined by the individual  $X_i$  is a linear ordering  $\preceq_i$  on the set  $Y$ . A *social aggregation rule* tries to fuse the set of linear orders  $\{\preceq_1, \dots, \preceq_n\}$  into a new one, say  $\preceq_S$  called *social*

*ordering* that tries to reflect somehow the main features of the individual orderings  $\preceq_1, \dots, \preceq_n$ . If  $R$  denotes the set of all the possible rankings or linear orders on  $Y$ , that can obviously be identified to  $S(Y)$ , namely the set of permutations of the elements of  $Y$ , a social aggregation rule, provided that  $X$  has  $n$  elements, will be a map from  $R^n$  into  $R$ .

We could go further, and suppose that each individual is a potential voter in a poll. Each voter has to define a whole ranking on the elements of  $Y$ , that could actually be interpreted as the set of candidates. If we do not know a priori how many persons will finally vote since people in a society could refuse to give their opinion voting, in order to define a good *social aggregation rule* we should take into consideration a map  $F$  from  $\mathcal{R} = \bigcup_{n=1}^{\infty} R^n$  into  $R$ .

Obviously,  $F$  should also accomplish some conditions, that perhaps could be understood as “*common sense restrictions*”. Among them, quite probably anonymity-neutrality and unanimity will actually be imposed to  $F$ .

More or less this is the theoretical setting of the Arrovian models arising in the 1950’s, that gave rise to the famous *Arrow’s impossibility theorem* in Social Choice (see [2, 42, 14]). Using mainly combinatorial techniques, Kenneth J. Arrow proved in 1951 that under a few, mild and apparently “common sense” restrictions, no social aggregation rule exists.

Since the appearance of this impossibility results, some other alternative models (e.g.: the ones introduced by Gibbard and Satterthwaite, see [33, 50]) using different sets of “common sense restrictions” were also considered in the specialized literature, mainly in the 1970’s. However, they also lead to impossibility results.

Then, in the early 1980’s new models based on topological considerations were introduced by G. Chichilnisky and G. Heal (see e.g. [20, 21]). The means in these model are called *topological means* and just use the conditions of anonymity-neutrality and unanimity, plus an extra condition of continuity as regards a given topology (see Definition 3.37 above). Unlike the previous models, these ones could lead to *possibility* results, depending on the topologies considered. (For further information see e.g. [14]).

## 5.2. Ranking sets of objects

A typical problem arising in Decision Theory consists in the search for a suitable extension to the power set of a total order defined on a finite set  $X$ .

Extensions of a linear order from a finite set to its power set actually exist. Perhaps the most common and well-known is the *lexicographic* one. (For instance, if  $X = \{a, b\}$  and  $a \preceq b$ , the lexicographic order  $\preceq_l$  on the power set  $\mathcal{P}(X)$  is  $\emptyset \preceq_l \{a\} \preceq_l \{a, b\} \preceq_l b$ ).

However, in many contexts, the extension required should accomplish different “common sense” *criteria* stated a priori. This is typical, for instance, when a firm wants to hire new people -no matter how many new workers-. To do so, the firm puts an exam to the individuals or candidates. How can the

firm, based on the individual ranking of individuals that comes from the results got in the exam, compare different *sets* -with more than one element, in general- of candidates? This kind of problems were deeply studied in the 1980's in contexts of Mathematical Economics (see e.g., [39, 10]). There are results in this direction that show that under some mild and apparently "common sense" criteria, the required extension is *never* possible. Of course, this will depend on the criteria chosen, that actually are not mild, but too restrictive and leading to impossibility, instead. (For a further account, see e.g. [7]).

Sometimes some criterion reminds us a hidden idea of a mean. For instance, a typical criterion is the following one:  $x \preceq y \Rightarrow \{x\} \preceq \{x, y\} \preceq \{y\} (x, y \in X)$ . Criteria of this kind (those that remind some sort of a mean) could be incompatible with criteria of another nature, so immediately giving rise to impossibility results. To put a trivial example, let us consider a criterion that depends on *cardinality* as  $\text{card}(A) < \text{card}(B) \Rightarrow A \prec B$  ( $A, B \subseteq X$ ). This criterion is obviously incompatible with the one shown above.

Moreover, we may also point out at this stage that extensions of a linear order on a finite set  $X$ , to its power set  $\mathcal{P}(X)$ , could also derive from suitable universal means, according to Theorem 3.15 (see also Remark 3.26 (iii) above).

### 5.3. Fuzzy Set Theory

In Fuzzy Set Theory, initiated by L.A. Zadeh in 1965 (see [53, 49]), a *fuzzy set* of a universe  $X$  is defined as a function  $\mu_X : X \rightarrow [0, 1]$ , called the *membership function* (or the *indicator*) of the fuzzy set. Given an element  $t \in X$ ,  $\mu_X(t)$  is interpreted as the degree on which the element  $t$  belongs to the fuzzy set  $X$ . Unlike the classical (also called *crisp*) set theory, where the *characteristic function*  $\chi$  of a subset  $X$  of a universe  $U$  takes values in  $\{0, 1\}$  so that either an element  $t \in U$  belongs to  $X$  if  $\chi(t) = 1$  or it does not belong when  $\chi(t) = 0$ , now the idea of set membership is *graduable* between 0 and 1, both included, by means of the indicator  $\mu_X$ .

In order to deal with fuzzy systems, making operations with them, and having in mind an idea of *aggregation* of several fuzzy sets on the same universe into a new one, it is typical to use  $n$ -dimensional operators  $F_n$  from the unit cube  $[0, 1]^n$  into the unit interval  $[0, 1]$ . In this way, if we are given  $n$  different fuzzy sets  $\{X_1, \dots, X_n\}$ , whose corresponding indicators are  $\{\mu_{X_1}, \dots, \mu_{X_n}\}$ , we may build a new fuzzy set  $Y$ , that fuses or aggregates the given fuzzy sets  $X_1, \dots, X_n$ , through the membership function  $\mu_Y$  given by  $\mu_Y(t) = F_n(\mu_{X_1}(t), \dots, \mu_{X_n}(t))$  for all  $t \in U$ .

Among classical operators in this theory we may consider triangular norms, triangular conorms and copulas (see e.g. [32, 43, 3, 37, 36, 51, 35]).

Thus, in particular, the following well-known definition is often encountered in this setting:

**Definition 5.1.** A (*bidimensional*) *triangular norm* is defined as a map  $T : [0, 1]^2 \rightarrow [0, 1]$  satisfying the following properties:

- (i)  $T(x, 1) = x$ , for every  $x \in [0, 1]$ .



- (ii)  $T(x, y) = T(y, x)$ , for every  $x, y \in [0, 1]$ .
- (iii) If  $x \leq x'$  then  $T(x, y) \leq T(x', y)$ , for every  $x, x', y \in [0, 1]$ .
- (iv)  $T$  is a continuous map.

Perhaps surprisingly, most of the typical operators in this framework are defined for the *bidimensional* case, that is, as functions from the unit plane  $[0, 1]^2$  into the unit interval  $[0, 1]$ . There are some exceptions, as for instance, the  $n$ -dimensional copulas analyzed in [51]. But these are scarce if compared when the common use of bidimensional aggregation operators.

We may wonder why this happens, that is, why there is a general lack of definitions of high dimensional operators and a totally common general use of bidimensional ones. At this stage, a possible answer comes to our mind: the key is *iterativity*. An attempt to define higher dimensional operators that generalize, say, triangular norms, could lead to iterative “means” that can be expressed iteratively in terms of bidimensional ones, in the spirit of the results introduced here in Section 4 (and Theorem 4.7 in particular). To see a clear example of this situation, suppose that we define tridimensional means, and let us see what happens.

**Definition 5.2.** A *tridimensional triangular norm* is defined as a map  $T : [0, 1]^3 \rightarrow [0, 1]$  satisfying the following properties:

- (i)  $T(x, 1, 1) = x$ , for every  $x \in [0, 1]$ .
- (iii)  $T$  satisfies the anonymity-neutrality condition.
- (iii)  $T$  satisfies a generalized condition of associativity, namely it holds true that  $T(T(x, y, z), u, v) = T(x, T(y, z, u), v) = T(x, y, T(z, u, v))$  for every  $x, y, z, u, v \in [0, 1]$ .
- (iv) If  $x \leq x'$  then  $T(x, y, z) \leq T(x', y, z)$ , for every  $x, x', y, z \in [0, 1]$ .
- (v)  $T$  is a continuous map.

In [12], Proposition 2.8, the following overwhelming result appears.

**Theorem 5.3.** *If  $T$  is a tridimensional triangular norm, there exists a (bidimensional) triangular norm  $F$  such that  $T(x, y, z) = F(F(x, y), z)$ , for every  $x, y, z \in [0, 1]$ .*

**Remark 5.4.** Notice that in the definition of triangular norms and conorms no condition of unanimity as  $T(x, x) = x$  has been given a priori. Observe also that the existence of the neutral element 1 in the case of triangular norms, since  $T(x, 1) = x$ , for every  $x \in [0, 1]$  (and  $T(x, 1, 1) = x$ , for every  $x \in [0, 1]$  if we consider tridimensional norms as defined before). This fact is a key for iterativity, more or less mimicking the steps of Theorem 4.7 (see also [12] for further details).

## 6. Discussion

Due to the wide set of contexts where some idea of a “mean” has been introduced, the task of adopting one general abstract definition is not easy, and can actually lead to controversy. Throughout this manuscript we have tried to find some “common factors” in most of the typical definitions. We have started by considering means that could directly act on any amount of elements. This is the reason why, beginning with a nonempty set  $X$ , we pass to consider functions directly defined on  $\mathcal{X} = \bigcup_{n=1}^{\infty} X^n$  and taking values in  $X$ , instead of working differently with maps from  $X^n$  into  $X$ , with  $n$  varying. Then we have considered the classical Kolmogorov’s setting in the real line, analyzing the restrictions involved. We have seen that the condition of strict monotonicity is enormously restrictive. It can by no means be adapted to other situations, as, e.g., finite totally ordered sets. Thus, we have then considered means defined on totally ordered sets, but dealing with monotonicity instead of strict monotonicity. Further, we have considered means on lattices. And this gives rise to the consideration of means on semigroups, too. Other miscellaneous kinds of means, defined on topological spaces or on nonempty sets endowed with some algebraic operation and structure have been defined, too. Looking at what all these concepts have in common, we finally introduce the condition of a *general mean* on a nonempty set. Different characterizations, and/or equivalent definitions in each particular case, have been obtained. We have mainly used techniques based on *functional equations*.

## 7. Conclusion

Bearing in mind the idea of *computability* when reckoning means, we have analyzed in depth the concept of iterativity, and some other related items as, e.g., associativity. This has also been made, mainly, with techniques of functional equations. In particular, we have proved that classical means as the Kolmogorov’s ones in the real line give rise to bad news from the point of view of algorithmic computability and iterativity. Again, this could be a reason to go further than just considering Kolmogorov’s means on the real line (or on real intervals), and, consequently, to explore many other possibilities.

In addition, and perhaps as a by-product, some explanation about why in some contexts only bidimensional aggregations are often defined (instead of  $n$ -dimensional ones for any  $n \in \mathbb{N}$ ) has been outlined. The key here is again iterativity, so that from bivariate maps we can reach  $n$ -bivariate ones following a standard iterative process. A typical important case is that of triangular norms on the unit interval.

There is a wide room for further research studies. To put only one example, in next future we could try to analyze “means” in which the condition of anonymity-neutrality is dropped. In the particular case of the real line, and perhaps working “à la Nagumo-Kolmogorov”, this could lead to characterizations of, say, quasi-linear means (see e.g., [5]). A possible application of this setting could arise in theoretical computer sciences, when different operations made by

a computer can be aggregated in a way, but not permuted, since the order in which they are performed could be relevant and in order to start some operation perhaps some others should compulsorily be finished. This could give rise to different new definitions related to the concepts we have introduced throughout the paper. To put just one example, in order to deal with aggregations of fuzzy sets, and perhaps dropping the anonymity-neutrality conditions, concepts as left-recursivity and right-recursivity have been defined and analyze (see e.g. [22, 4, 13, 35]). These conditions would agree with iterativity under anonymity-neutrality. They give also rise to functional equations that lie in the orbit of associativity (see e.g. [46]).

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